Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

REMARKS ON SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Professor of Mathematics University of Michigan, Ann Arbor, Michigan.

Introductory. "The mathematics of the twenty-first century may be very different from our own; perhaps the school boy will begin algebra with the theory of substitution groups as he might now but for inherited habits."[*]

These words imply two important features of substitution groups, viz., their extensive application and their rudimentary character, for a beginners' algebra must in all times possess these features. To these we may perhaps add a third, viz., their usefulness for unification; for we can scarcely expect that much will be added to the beginners' algebra except what tends to show the unity of subjects which otherwise appear distinct and thus to simplify the acquisition of a knowledge of them. The last one of these features is especially prominent in substitution groups and on this account it seems probable that they will increase in favor with the progress of thought towards abstraction and generalization.

The theory of substitution groups does not concern itself with metrical properties nor is it a part of the science of quantity in the ordinary sense. It chiefly investigates correspondencies by means of which we may apply results with respect to one subject to another without going through the work of investigating each seperately. It tends therefore to save thought and thus becomes a new instrument in the hands of the student by means of which he can work not only faster but also with less exertion than he could without employing it. With these general statements in regard to the scope and features of our subject we proceed to some special consideration of its nature and a few important applications. In doing this we shall not presuppose any knowledge of the subject on the part of the reader.

If we take the four numbers $1, -1, \sqrt{-1}, -\sqrt{-1}$ and multiply any one by itself or by another one of them we obtain no new numbers; thus

$$-\sqrt{-1} \times -\sqrt{-1} = -1,\ -\sqrt{-1} \times -1 = \sqrt{-1}$$
$$-\sqrt{-1} \times \sqrt{-1} = 1,\ -1 \times -1 = 1, \text{ etc.}$$

We therefore say that these four numbers form a *group* with respect to multiplication. If we take 1 and operate upon it by itself we obtain no new number. 1 therefore forms a *subgroup*[†] of the given group. If we take -1 in a similar way we obtain the subgroup $1, -1$ since $(-1)^2 = 1, (-1)^3 = -1$, etc. By taking either $\sqrt{-1}$ or $-\sqrt{-1}$ in this way we will obtain the entire group, since $(\sqrt{-1})^2 = -1, (\sqrt{-1})^3 = -\sqrt{-1}, (\sqrt{-1})^4 = 1, (\sqrt{-1})^5 = \sqrt{-1}$, etc. We say therefore that the above group is *generated* by either $\sqrt{-1}$ or $-\sqrt{-1}$ and that $1, -1$ generate the subgroups 1 and $(1, -1)$ respectively.

---

[*] From Professor Simon Newcomb's address delivered before the New York Mathematical Society at its annual meeting, December 28, 1893, and published in the *Bulletin* of this society, vol. III., No. 4.

[†] The terms group and subgroup are only relative. If a subgroup is considered by itself it is called a group and a group may in turn be a subgroup of a larger group.

We may now consider the corresponding *substitution* group

$$1,\ ac.bd,\ abcd,\ adcb.$$

Where $ac.bd$ means that c is substituted for a and a for c, then d for b and b for d; $abcd$ means that b is substituted for a, c for b, d for c and a for d. 1 means that all the letters are replaced by themselves, or, what amounts to the same, that the letters all remain unchanged.

For example, if we operate upon $a+2b+3c+4d$ with $ac.bd$ we obtain $c+2d+3a+4b$ and if we operate upon the same expression with $abcd$ and $adcb$ we obtain $b+2c+3d+4a$, $d+2a+3b+4c$ respectively. From these definitions it follows directly that 1, $ac.bd$ generate the subgroups 1 and $(1, ac.bd)$ respectively, for if we perform the operation indicated by these symbols any number of times and in any order we obtain a result which is equivalent to the operation indicated by one of the symbols in these subgroups. This is the necessary and sufficient condition that a given aggregate of symbols may be a *group* or subgroup. Since $abcd^2$* $=ac.bd$, $abcd^3=adcb$, $abcd^4=1$, $abcd^5=abcd$, etc., it follows that the above group is generated by either of the substitutions $abcd$ or $adcb$.

In the study of substitution groups it is very important to be familiar with quite a number of such groups. We therefore shall give all the groups of the different numbers of letters not exceeding four. It is evident that the only substitution group of two letters is

$$1,\ ab.$$

The following are the only two that involve three letters:

1	abc	ab
	acb	ac
		bc

1	abc
	acb

The following seven are those that involve four letters:

1	abc	$abcd$	ac	$ab.cd$
	acb	$adcb$	ab	$ac.bd$
	abd	$acbd$	ad	$ad.bc$
	adb	$adbc$	bc	
	acd	$abdc$	bd	
	adc	$acdb$	cd	
	bcd			
	bdc			

1	abc	$ab.cd$
	acb	$ac.bd$
	abd	$ad.bc$
	adb	
	acd	
	adc	
	bdc	
	bdc	

1	$ab.cd$	$abcd$	ac
	$ac.bd$	$adcb$	bd
	$ad.bc$		

1	$ab.cd$
	$ac.bd$
	$ad.bc$

1	$abcd$	$ac.bd$
	$adcb$	

1	ac	$ac.bd$
	bd	

1	$ac.bd$

---

* This means that the operation indicated by $abcd$ is to be performed twice in succession. In general $abcd^x=abcd.abcd\ldots.abcd$. the operation $abcd$ being performed x times in succession. If we operate once upon $a+2b+3c+4d$ with $abcd$ we obtain $b+2c+3d+4a$, on operating again we obtain $c+2d+3a+4b$; the last result would have been obtained by operating with $ac.bd$ upon the first expression.

The largest group of any given number of letters contains all the substitutions which correspond to the permutations of these letters. The number of substitutions in this group is therefore $n!$ and all the other groups of the same number of letters are subgroups of it. The second group, with respect to size contains all the substitutions which correspond to an even number of interchanges of letters, the number of substitutions in this group is $n!\div 2$.

[To be continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the April Number.]

PROPOSITION XVII. *In the hypothesis of acute angle, we can find a perpendicular and an oblique to the same straight which never meet.*

More literally: *If the straight AH stands* (fig. 15.) *at right angles to any one certain straight AB however small: I say that in the hypothesis of acute angle it cannot hold good, that every straight BD, making with AB any acute angle you choose, toward the parts of this AH, will at length meet this AH produced at a finite, or terminated distance.*

PROOF. Join HB. The angle ABH will be acute (Eu. I. 17), because of the right angle at the point A. Now (Eu. I. 23) a certain HD can be drawn toward the parts of the point B, which not cutting the angle AHB makes with this HB an acute angle equal to this acute angle ABH. Then from the point B is let fall to HD the perpendicular BD, which falls toward the parts of the aforesaid acute angle at the point H. Since therefore the side HB is opposite in the triangle HDB to the right angle at D, and likewise in the triangle BAH to the right angle at A; and again in those two triangles equal angles are adjacent to this side HB, which are in the first triangle indeed the angle BHD, and in the latter the angle HBA; also (Eu. I. 26) the remaining angle HBD in the former triangle will be equal to the remaining angle BHA in the latter triangle. Wherefore the entire angle DBA will be equal to the entire angle AHD. Now however, neither of the aforesaid equal angles will be obtuse, lest we meet (from the preceding proposition) in one case the now rejected hypothesis of obtuse angle. Nor will

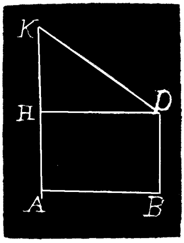

Fig. 15.